Contributions on the Extension of the Optimal Homotopy Asymptotic Method in Solution of the Flow of the Polymeric Materials

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An incompressible MHD flow of two-dimensional upper-convected Maxwell fluid over a porous stretching plate with suction is studied. The nonlinear differential equation is solved approximately by means of the Optimal Homotopy Asymptotic Method (OHAM). Multiple solutions are given, showing a very good agreement between the analytical and numerical solutions. This procedure is very efficient in practice, ensuring a very rapid convergence of the solutions after only one iteration.

Keywords: Maxwell fluid, porous stretching plate, optimal homotopy asymptotic method

To obtain the plastic materials from the polymers by means of injection procedure, it is necessary to know well determined flow conditions. From the mathematical point of view, the flow problem can be solved with Optimal Homotopy Asymptotic Method in the conditions presented in this work.

In practical applications, non-Newtonian fluids are more appropriate than Newtonian fluids and therefore, the flows of non-Newtonian fluids have been analyzed by numerous researchers. Examples of the flow of non-Newtonian fluids occur in a large variety of applications: plastic polymers, synthetic fibres, drilling muds and so on. One type of fluids in which the relaxation type phenomena can be considered is known as Maxwell model. Some investigations in this field are made by the engineers, physicians and computer scientists. Sakiadis [1] first studied various aspects of the stretching problem: the flow due to a semi-infinite horizontally moving wall in an ambient fluid. Phan-Thieu [2] and Zheng et al. [3] considered the plane and axisymmetric stagnation flows in a Maxwell fluid using the shooting and boundary element method. Sadeghy et al. [4] considered the problem of hydrodynamic Sakiadis flow of an upper-convected Maxwell fluid over a rigid plate moving steadily in an otherwise quiescent fluid. Homotopy analysis method is used by Hayat et al. [5] to solve non-linear differential equation of the upper-convected Maxwell fluid. Sahoo [6] investigated the effects of partial slip on the MHD flow and mass transfer of an electrically conducting second grade fluid past an axisymmetric stretching sheet.

The objective of the present paper is to propose an accurate procedure to nonlinear differential equation of the magnetohydrodynamic flow problem of an upper-convected Maxwell fluid over a porous stretching plate using OHAM. A version of the OHAM is applied in this study to derive highly accurate analytical expressions of the solutions. The main advantage of this approach is the control of the convergence of approximate solutions in a very rigorous way. A very good agreement was found between our approximate solutions and numerical results, which proves that our method is very efficient in practice and accurate.

Equation of motion

If we consider the steady, incompressible, two-dimensional flow of an upper-convected Maxwell fluid over a porous stretching plate, by imposing an uniform magnetic field B_0 along the y - direction and neglecting the induced magnetic field, the equations which govern the steady flow can be written in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}-\delta B_0^2 u \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\partial S_{yx}}{\partial x}+\frac{\partial S_{yy}}{\partial y} \tag{3}$$

where *u, v* are the velocity components, ρ is the density, δ is the electrical conductivity and S_{xx}, S_{xy}, S_{yx}, S_{yy} are the components of the extra tensor *S*.

Using the boundary layer approximations [4]

$$u=O(1),\ v=O(\delta),\ x=O(1),\ y=O(\delta) \tag{4}$$

$$\frac{T_{xx}}{\rho}=O(1),\ \frac{T_{xy}}{\rho}=O(\delta),\ \frac{T_{yy}}{\rho}=O(\delta^2) \tag{5}$$

the flow in the absence of the pressure gradient is governed by eq. (1) and

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+\lambda\left(u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right)=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\delta B_0^2}{\rho}u \tag{6}$$

where λ is the relaxation time and ν is the kinematic viscosity of fluid.

The relevant initial/boundary conditions for the flow-problem are

$$\begin{aligned}&u=Cx,\ v=-V_0,\ at\ y=0\\&u\to 0,\ as\ y\to\infty\end{aligned} \tag{7}$$

in which *C* is the stretching rate, and $V_o > 0$ is the suction velocity.

Introducing the stream function ψ such that

$$u=\frac{\partial\Psi}{\partial y},\quad v=-\frac{\partial\Psi}{\partial x} \tag{8}$$

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then eq. (1) is identically satisfied and eq. (6) becomes

$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2}+\lambda[(\frac{\partial\Psi}{\partial y})^2\frac{\partial^3\Psi}{\partial x^2\partial y}+(\frac{\partial\Psi}{\partial x})^2\frac{\partial^3\Psi}{\partial y^3}-$$
$$-2\frac{\partial\Psi}{\partial y}\frac{\partial\Psi}{\partial x}\frac{\partial^3\Psi}{\partial x\partial y^2}]=\nu\frac{\partial^3\Psi}{\partial y^3}-\frac{\delta B_0^2}{\rho}\frac{\partial\Psi}{\partial y} \quad (9)$$

Using the similarity transformations

$$\eta=\sqrt{\frac{C}{\nu}}y,\quad \Psi=\sqrt{\nu C}xf(\eta) \quad (10)$$

then, eq. (8) become

$$u=Cxf'(\eta),\quad v=-\sqrt{\nu C}f(\eta). \quad (11)$$

The governing equation is obtained substituting eq. (11) into eq. (9):

$$f'''-M^2f'-f'^2+ff''+\beta(2ff'f''-f^2f''')=0 \quad (12)$$

where $M^2=\frac{\delta B_0^2}{\rho C}$ and $\beta=\lambda C$.

The initial/boundary conditions become:

$$f(0)=R,\quad f'(0)=1,\quad f'(\infty)=0 \quad (13)$$

with $R=\frac{V_0}{\sqrt{\nu C}}>0$.

In the following, the nonlinear differential equation (12) with initial/boundary conditions (13) can be solved using OHAM.

Basic ideas of the optimal homotopy asymptotic method

Eq. (12) with initial / boundary conditions (13) can be written in a more general form

$$N[f(\eta)]=0 \quad (14)$$

where N is a given nonlinear differential operator depending on the unknown function f(η), subject to the initial / boundary conditions

$$B\left(f(\eta),\frac{df(\eta)}{d\eta}\right)=0. \quad (15)$$

Let $f_o(\eta)$ be an initial approximation of f(η) and L an arbitrary linear operator such as

$$L[f_0(\eta)]=0,\ B\left(f_0(\eta),\frac{df_0(\eta)}{d\eta}\right)=0. \quad (16)$$

It should be emphasized that this linear operator L is not unique.

If p ∈ [0, 1] denotes an embedding parameter and F is an analytic function, then we propose to construct a homotopy [7-11]:

$$\mathcal{H}[L(F(\eta,p)),H(\eta,C_i),N(F(\eta,p))],\ i=1,2,...,s \quad (17)$$

with the properties

$$\mathcal{H}[L(F(\eta,0)),H(\eta,C_i),N(F(\eta,0))]=L(F(\eta,0))=L(f_0(\eta))=0 \quad (18)$$

$$\mathcal{H}[L(F(\eta,1)),H(\eta,C_i),N(F(\eta,1))]=H(\eta,C_i)N(F(\eta))=0,\ i=1,2,...,s \quad (19)$$

where $H(\eta, C_i) \neq 0$ is an arbitrary auxiliary convergence-control function depending on variable η and on a number of arbitrary parameters $C_1, C_2,...,C_s$ unknown now and will be determined later.

Let we consider the function F in the form

$$F(\eta,p)=f_0(\eta)+pf_1(\eta,C_i). \quad (20)$$

By substituting eq. (20) into equation obtained by means of homotopy (17)

$$\mathcal{H}[L(F(\eta,p)),H(\eta,C_i),N(F(\eta,p))]=0,\ i=1,...,s \quad (21)$$

and then equating the coefficients of p^o and p^1, we obtain:

$$\mathcal{H}[L(F(\eta,p)),H(\eta,C_i),N(F(\eta,p))]=$$
$$=L(f_0(\eta))+p[L(f_1(\eta,C_i))-L(f_0(\eta))-H(\eta,C_i)N(f_0(\eta))],\ i=1,2,...,s. \quad (22)$$

From eq. (22) we obtain the governing equation of $f_o(\eta)$ given by eq. (16) and the governing equation of $f_1(\eta)$, i.e.

$$L(f_1(\eta,C_i))=H(\eta,C_i)N(f_0(\eta)),\quad B\left(f_1(\eta,C_i),\frac{df_1(\eta,C_i)}{d\eta}\right)=0,\ i=1,...,s \quad (23)$$

where we find the following expression for the nonlinear operator:

$$N(f_0(\eta))=\sum_{i=1}^{m}h_i(\eta)g_i(\eta) \quad (24)$$

where the functions $h_i(\eta)$ and $g_i(\eta)$, i = 1,..., m are known and depend on the function $f_o(\eta)$ and also on the nonlinear operator, m being a known integer number.

In this way, taking into account eq. (19), from eq. (20) for p=1, we obtain the first-order approximate solution which becomes

$$\bar{f}(\eta,C_i)=f_0(\eta)+f_1(\eta,C_i),\quad i=1,...,s \quad (25)$$

It should be emphasized that $f_o(\eta)$ and $f_i(\eta, C_i)$ are governed by the linear eqs. (16) and (23) respectively with boundary conditions that come from the original problem. It is known that the general solution of nonhomogeneous linear eq. (23) is equal to the sum of general solution of the corresponding homogeneous equation and of some particular solutions of the nonhomogeneous equation. However, the particular solutions are readily selected only in the exceptional cases.

In what follows we do not solve eq. (23), but from the theory of differential equations, taking into considerations the method of variation of parameters, Cauchy method, method of influence function, the operator method [12] and so on, is more convenient to consider the unknown function $f_1(\eta)$, in the form

$$f_1(\eta,C_j)=\sum_{i=1}^{m}H_i(\eta,h_j(\eta),C_j)g_i(\eta),\quad j=1,...,s$$
$$B\left(f_1(\eta,C_i),\frac{df_1(\eta,C_i)}{d\eta}\right)=0 \quad (26)$$

where within expression of $H_i(\eta),h_j(\eta),C_j)$ appear linear combinations of some functions $h_j(a)$, the same terms which are given by the corresponding homogeneous equation and the unknown parameters C_j, j = 1,..., s. In the sum from eq. (26) appear an arbitrary number of *m* for such terms.

For instance if $h_i=\sin\alpha\eta$, then we can choose

$H_1(\eta,h_1,C_j)=C_1\sin\alpha\eta+C_2\cos\alpha\eta+C_3\sin 2\alpha\eta+\ldots$. Similarly, if $h_1=\eta^3$, then we can choose $H_1(\eta,h_1,C_j)=C_1\eta^3+C_2\eta+C_3\eta^2+C_4\eta^4+\ldots$. In the case when $h_1=\ln\eta$, we can choose $H_1(\eta,h_1,C_j)=C_1\ln\eta+C_2\ln^2\eta+C_3\eta\ln\eta+\ldots$ or $H_1(\eta,h_1,C_j)=C_1\ln\eta+C_2\eta\ln\eta+C_3\eta^2\ln\eta+C_4\eta\ln 2\eta+\ldots$. We have a large freedom to choose the value of *m*

We cannot demand that $f_i=(\eta, C_i)$ to be solutions of eq. (23) but $\bar{f}(\eta,C_i)$ given by eq. (25) with $f=(\eta, C_i)$ given by eq. (26), are the solutions of the eq. (14). This is underlying idea of our method. The convergence of the approximate solution $\bar{f}=(\eta, C_i)$ given by eq. (25) depends upon the auxiliary functions $H_1(\eta, h_i, C_j)$, j=1,...,s. There are many possibilities to choose these functions H_i. We try to choose H_i so that within eq. (26) the terms $\sum_{i=1}^{m}H_i(\eta,h_j(\eta),C_j)g_i(\eta)$ to be of the same shape with the terms $\sum_{i=1}^{m}h_i(\eta)g_i(\eta)$ given by eq. (24). The first-order approximate solution $\bar{f}(\eta,C_i)$ also depend on the parameters C_j, j = 1,...,s. The values of these parameters can be optimally identified via various methods, such as: the least-square method, the Galerkin method, the collocation method, the Ritz method, and so on. The first option should be minimizing the square residual error:

$$J(C_1,C_2,\ldots,C_s)=\int_{(D)}R^2(\eta,C_1,C_2,\ldots,C_s)\,d\eta \quad (27)$$

where the residual *R* is given by

$$R(\eta,C_1,C_2,\ldots,C_s)=N\left(\bar{f}(\eta,C_i)\right) \quad (28)$$

The unknown parameters C_1, C_2,...C_s can be identified from the conditions:

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\ldots=\frac{\partial J}{\partial C_s}=0. \quad (29)$$

With these parameters known (called optimal convergence-control parameters), the first-order approximate solution given by eq. (25) is well-determined.

It should be emphasized that our procedure contains the auxiliary functions $H_i(\eta, f_i, C_j)$, i=1,...,m, j=1,...,s, which provides us with a simple way to adjust and control the convergence of the approximate solutions. It is very important to properly choose these functions $H_i(\eta, f_i, C_j)$, which appear in the construction in the first-order approximation.

Multiple solutions for the upper-convected Maxwell fluid with OHAM

We apply our procedure to obtain approximate solutions of eqs. (12) and (13). We choose the linear operator of the form

$$L[f(\eta)]=f'''(\eta)+\frac{3K}{K\eta+1}f''(\eta). \quad (30)$$

We mention that the linear operator is not unique. Also, we have freedom to choose:

$$L[f(\eta)]=f'''(\eta)-\frac{6K^2}{(K\eta+1)^2}f'(\eta). \quad (31)$$

$$L[f(\eta)]=f'''(\eta)+Kf''(\eta). \quad (32)$$

$$L[f(\eta)]=f'''(\eta)-K^2f'(\eta). \quad (33)$$

here K is an unknown positive parameter and will be determined later.

From eq. (16) with the initial/boundary conditions

$$f_0(0)=R,\quad f_0'(0)=1,\quad f_0'(\infty)=0. \quad (34)$$

we can obtain the initial approximation in the form:

$$f_0(\eta)=R+\frac{1}{K}-\frac{1}{K(K\eta+1)} \quad (35)$$

where we used the linear operator given by eq. (30). The nonlinear operator corresponding to nonlinear differential eq. (12) is defined as

$$N[f(\eta)]=f'''-M^2f'-f'^2+ff''+\beta(2ff'f''-f^2f'''). \quad (36)$$

Substituting eq. (35) into eq. (36) it holds that

$$N[f_0(\eta)]=-\frac{M^2}{(K\eta+1)^2}-\frac{2(1+KR)}{(K\eta+1)^3}+ +\frac{1+6K^2-6\beta(1+KR)^2}{(K\eta+1)^4}-\frac{16\beta(1+KR)}{(K\eta+1)^5}+\frac{10\beta}{(K\eta+1)^6} \quad (37)$$

Comparing eqs. (24) and (37) one can get

$$\begin{aligned}&h_1(\eta)=-M^2,\ h_2(\eta)=-2(1+KR),\\ &h_3(\eta)=1-6K^2-6\beta(1+KR)^2,\ h_4(\eta)=-16\beta(1+KR),\ h_5(\eta)=10\beta,\\ &g_1(\eta)=\frac{1}{(K\eta+1)^2},\ g_2(\eta)=\frac{1}{(K\eta+1)^3},\ g_3(\eta)=\frac{1}{(K\eta+1)^4}\\ &g_4(\eta)=\frac{1}{(K\eta+1)^5},\ g_5(\eta)=\frac{1}{(K\eta+1)^6}.\end{aligned} \quad (38)$$

The function $f_1(\eta)$ given by eq. (26) becomes

$$f_1(\eta,C_i)=H_1(\eta,C_i)\frac{1}{(K\eta+1)^2}+H_2(\eta,C_i)\frac{1}{(K\eta+1)^3}+\ldots+ +H_j(\eta,C_i)\frac{1}{(K\eta+1)^{j+1}}$$

$$f_1(0,C_i)=f_1'(0,C_i)=f_1'(\infty,C_i)=0 \quad (39)$$

where we have freedom to choose a lot of possibilities for the unknown functions H_i, i = 1,..., j as follows:

If we choose $H_i = C_i\eta^2$ and j=9, i=1,...,9, one can get

$$\begin{aligned}f_1(\eta,C_i)=&C_1\frac{\eta^2}{(K\eta+1)^2}+C_2\frac{\eta^2}{(K\eta+1)^3}+\\ &+C_3\frac{\eta^2}{(K\eta+1)^4}+C_4\frac{\eta^2}{(K\eta+1)^5}+\\ &+C_5\frac{\eta^2}{(K\eta+1)^6}+C_6\frac{\eta^2}{(K\eta+1)^7}+C_7\frac{\eta^2}{(K\eta+1)^8}+\\ &+C_8\frac{\eta^2}{(K\eta+1)^9}+C_9\frac{\eta^2}{(K\eta+1)^{10}}\end{aligned} \quad (40)$$

$$\overline{f}(\eta,C_i)=R+\frac{1}{K}-\frac{1}{K(K\eta+1)}+C_1\frac{\eta^2}{(K\eta+1)^2}+C_2\frac{\eta^2}{(K\eta+1)^3}+C_3\frac{\eta^2}{(K\eta+1)^4}+$$
$$+C_4\frac{\eta^2}{(K\eta+1)^5}+C_5\frac{\eta^2}{(K\eta+1)^6}+C_6\frac{\eta^2}{(K\eta+1)^7}+C_7\frac{\eta^2}{(K\eta+1)^8}+C_8\frac{\eta^2}{(K\eta+1)^9}+C_9\frac{\eta^2}{(K\eta+1)^{10}} \quad (41)$$

For $H_i = C_i\eta^2$ and j =10, i = 1, ..., 10, the first=order approximate solution becomes

$$\overline{f}(\eta,C_i)=R+\frac{1}{K}-\frac{1}{K(K\eta+1)}+C_1\frac{\eta^2}{(K\eta+1)^2}+C_2\frac{\eta^2}{(K\eta+1)^3}+C_3\frac{\eta^2}{(K\eta+1)^4}+C_4\frac{\eta^2}{(K\eta+1)^5}+$$
$$+C_5\frac{\eta^2}{(K\eta+1)^6}+C_6\frac{\eta^2}{(K\eta+1)^7}+C_7\frac{\eta^2}{(K\eta+1)^8}+C_8\frac{\eta^2}{(K\eta+1)^9}+C_9\frac{\eta^2}{(K\eta+1)^{10}}+C_{10}\frac{\eta^2}{(K\eta+1)^{11}}$$

If $H_i = C_i\eta^2$ and j =11, i = 1, ..., 11, we have (42)

$$\overline{f}(\eta,C_i)=R+\frac{1}{K}-\frac{1}{K(K\eta+1)}+C_1\frac{\eta^2}{(K\eta+1)^2}+C_2\frac{\eta^2}{(K\eta+1)^3}+C_3\frac{\eta^2}{(K\eta+1)^4}+$$
$$+C_4\frac{\eta^2}{(K\eta+1)^5}+C_5\frac{\eta^2}{(K\eta+1)^6}+C_6\frac{\eta^2}{(K\eta+1)^7}+C_7\frac{\eta^2}{(K\eta+1)^8}+C_8\frac{\eta^2}{(K\eta+1)^9}+ \quad (43)$$
$$+C_9\frac{\eta^2}{(K\eta+1)^{10}}+C_{10}\frac{\eta^2}{(K\eta+1)^{11}}+C_{11}\frac{\eta^2}{(K\eta+1)^{12}}$$

The first-order approximate solution given by eq. (25) is obtained from eqs. (35) and (39)
It is clear that in this way, we can obtain many other solutions.

Numerical results

We illustrate the accuracy of our procedure comparing results obtained through our procedure with numerical results. Optimal convergence-control parameters C_i are determined by means of the least-square method, using Wolfram Mathematica 6.0 software.

For $\beta = 0.5$, R = 0.25 and M = 0.75 for every case, we obtain the following results:

For the first-order approximate solution (41), one get:

$$\overline{f}(\eta)=1.7640282387-\frac{1.5140282387}{1+0.6604896622x}-\frac{0.4038516600x^2}{(1+0.6604896622x)^2}-\frac{0.2651317383x^2}{(1+0.6604896622x)^3}+$$
$$+\frac{1.5875750484x^2}{(1+0.6604896622x)^4}-\frac{9.9830883215x^2}{(1+0.6604896622x)^5}+\frac{40.2722023404x^2}{(1+0.6604896622x)^6}-\frac{81.8453842441x^2}{(1+0.6604896622x)^7}+$$
$$+\frac{89.8542653196x^2}{(1+0.6604896622x)^8}-\frac{51.6009371221x^2}{(1+0.6604896622x)^9}+\frac{12.2170946834x^2}{(1+0.6604896622x)^{10}} \quad (44)$$

The approximate solution (42) becomes

$$\overline{f}(\eta)=1.5246445734-\frac{1.2746445734}{1+0.7845324263x}-\frac{0.4232066131x^2}{(1+0.7845324263x)^2}-\frac{0.2248813142x^2}{(1+0.7845324263x)^3}+$$
$$+\frac{2.7575287469x^2}{(1+0.7845324263x)^4}-\frac{20.1233470298x^2}{(1+0.7845324263x)^5}+\frac{100.1896168227x^2}{(1+0.7845324263x)^6}-\frac{269.3616538738x^2}{(1+0.7845324263x)^7}+ \quad (45)$$
$$+\frac{418.6971099614x^2}{(1+0.7845324263x)^8}-\frac{382.2371075125x^2}{(1+0.7845324263x)^9}+\frac{191.2755083010x^2}{(1+0.7845324263x)^{10}}-\frac{40.5986791066x^2}{(1+0.7845324263x)^{11}}$$

In the last case, the first-order approximate solution (43) is written as

$$\overline{f}(\eta)=3.3822265399-\frac{3.1322265399}{1+0.3192617096x}-\frac{0.2570668647x^2}{(1+0.3192617096x)^2}-\frac{0.2781821246x^2}{(1+0.3192617096x)^3}+$$
$$+\frac{0.7344770160x^2}{(1+0.3192617096x)^4}-\frac{5.8351903615x^2}{(1+0.3192617096x)^5}+\frac{23.9865472994x^2}{(1+0.3192617096x)^6}-$$
$$-\frac{65.8073451539x^2}{(1+0.3192617096x)^7}+\frac{118.7010136318x^2}{(1+0.3192617096x)^8}-\frac{136.7897035346x^2}{(1+0.31926170960x)^9}+$$

$$+\frac{97.2920818266x^2}{(1+0.3192617096x)^{10}}-\frac{39.0230935985x^2}{(1+0.3192617096x)^{11}}+\frac{6.7652417220x^2}{(1+0.3192617096x)^{12}} \quad (46)$$

In figures 1 and 2 are plotted a comparison between the first-order approximate solutions and numerical results obtained by means of the fourth-order Runge-Kutta method in combination with the shooting method.

In tables 1-6 we present a comparison between the first-order approximate solutions $\bar{f}$ and f' given by eqs. (44 - 46) respectively, with numerical results for the same values of variable η.

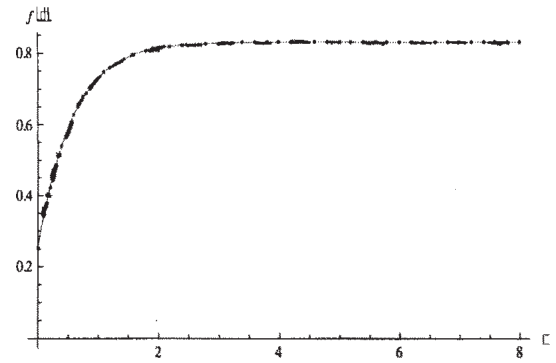

Fig. 1 Comparison between the approximate solution eq. (44) and numerical solution in the case β = 0.5, M=0.75, R=0.25: ———— numerical solution; ………… approximate solution

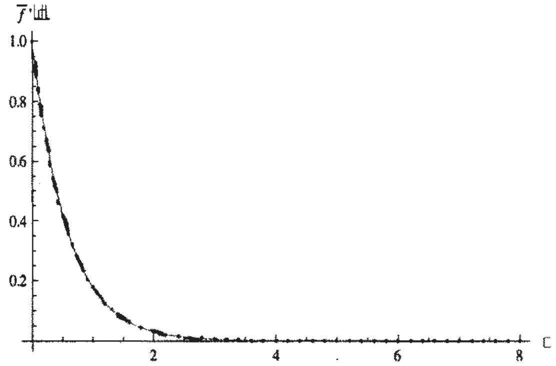

Fig. 2 Comparison between the derivative of the first-order approximate solution of the eq. (44) and numerical solution in the case β = 0.5, M=0.75, R=0.25: ———— numerical solution; .... approximate solution

Table 1

COMPARISON BETWEEN THE FIRST-ORDER APPROXIMATE SOLUTION $\bar{f}$ GIVEN BY EQ. (44) WITH NUMERICAL SOLUTION

η	$\bar{f}_{OHAM}(\eta)$ from Eq. (44)	$f_{numerical}(\eta)$	relative error $\varepsilon = \lvert f_{numerical}(\eta) - \bar{f}_{OHAM}(\eta) \rvert$
0	0.25	0.25	0.
4/5	0.6867508021	0.6867557534	$4.95 \cdot 10^{-6}$
8/5	0.7952864463	0.7952831549	$3.29 \cdot 10^{-6}$
12/5	0.8219220867	0.8219262745	$4.18 \cdot 10^{-6}$
16/5	0.8284650425	0.8284614200	$3.62 \cdot 10^{-6}$
4	0.8300669092	0.8300643067	$2.60 \cdot 10^{-6}$
24/5	0.8304557254	0.8304574546	$1.72 \cdot 10^{-6}$
28/5	0.8305528738	0.8305538923	$1.01 \cdot 10^{-6}$
32/5	0.8305813335	0.8305775608	$3.77 \cdot 10^{-6}$
36/5	0.8305918172	0.8305833895	$8.42 \cdot 10^{-6}$
8	0.8305950529	0.8305848529	$1.02 \cdot 10^{-5}$

Table 2

COMPARISON BETWEEN THE DERIVATIVE $\bar{f}'$ OBTAINED FROM EQ. (44) WITH NUMERICAL SOLUTION

η	$\bar{f}'_{OHAM}(\eta)$ from Eq. (44)	$f'_{numerical}(\eta)$	relative error $\varepsilon = \lvert f'_{numerical}(\eta) - \bar{f}'_{OHAM}(\eta) \rvert$
0	1.	1.	0.
4/5	0.2522893400	0.2523010529	$1.17 \cdot 10^{-5}$
8/5	0.0619955188	0.0620107896	$1.52 \cdot 10^{-5}$
12/5	0.0152177790	0.0152114260	$6.35 \cdot 10^{-6}$
16/5	0.0037367740	0.0037309432	$5.83 \cdot 10^{-6}$
4	0.0009090257	0.0009150971	$6.07 \cdot 10^{-6}$
24/5	0.0002215776	0.0002244562	$2.87 \cdot 10^{-6}$
28/5	0.0000593412	0.0000550683	$4.27 \cdot 10^{-6}$
32/5	0.0000202585	0.0000135305	$6.72 \cdot 10^{-6}$
36/5	$7.68 \cdot 10^{-6}$	$3.35 \cdot 10^{-6}$	$4.32 \cdot 10^{-6}$
8	$9.16 \cdot 10^{-7}$	$8.74 \cdot 10^{-7}$	$4.18 \cdot 10^{-8}$

Table 3

COMPARISON BETWEEN THE FIRST-ORDER APPROXIMATE SOLUTION $\bar{f}$ GIVEN BY EQ. (45) WITH NUMERICAL SOLUTION

η	$\bar{f}_{OHAM}(\eta)$ from Eq. (45)	$f_{numerical}(\eta)$	relative error $\varepsilon = \lvert f_{numerical}(\eta) - \bar{f}_{OHAM}(\eta) \rvert$
0	0.25	0.25	0.
4/5	0.6867549888	0.6867557534	$7.64 \cdot 10^{-7}$
8/5	0.7952862406	0.7952831549	$3.08 \cdot 10^{-6}$
12/5	0.8219231492	0.8219262745	$3.12 \cdot 10^{-6}$
16/5	0.8284633218	0.8284614200	$1.901 \cdot 10^{-6}$
4	0.8300665681	0.8300643067	$2.26 \cdot 10^{-6}$
24/5	0.8304565209	0.8304574546	$9.33 \cdot 10^{-7}$
28/5	0.8305529772	0.8305538923	$9.15 \cdot 10^{-7}$
32/5	0.8305798428	0.8305775608	$2.28 \cdot 10^{-6}$
36/5	0.8305889777	0.8305833895	$5.58 \cdot 10^{-6}$
8	0.8305917010	0.8305848529	$6.84 \cdot 10^{-6}$

η	$\bar{f}'_{OHAM}(\eta)$ from Eq. (45)	$f'_{numerical}(\eta)$	relative error $\varepsilon = \lvert f'_{numerical}(\eta) - \bar{f}'_{OHAM}(\eta) \rvert$
0	1.	1.	0.
4/5	0.2522730369	0.2523010529	$2.801 \cdot 10^{-5}$
8/5	0.0620076480	0.0620107896	$3.14 \cdot 10^{-6}$
12/5	0.0152121933	0.0152114260	$7.67 \cdot 10^{-7}$
16/5	0.0037366407	0.0037309432	$5.69 \cdot 10^{-6}$
4	0.0009113768	0.0009150971	$3.72 \cdot 10^{-6}$
24/5	0.0002217970	0.0002244562	$2.65 \cdot 10^{-6}$
28/5	0.0000576356	0.0000550683	$2.56 \cdot 10^{-6}$
32/5	0.0000182306	0.0000135305	$4.70006 \cdot 10^{-6}$
36/5	$6.45 \cdot 10^{-6}$	$3.35 \cdot 10^{-6}$	$3.102 \cdot 10^{-6}$
8	$8.68 \cdot 10^{-7}$	$8.74 \cdot 10^{-7}$	$6.39 \cdot 10^{-9}$

Table 4
COMPARISON BETWEEN THE DERIVATIVE $\overline{f'}$ OBTAINED FROM EQ. (45) WITH NUMERICAL SOLUTION

η	$\bar{f}_{OHAM}(\eta)$ from Eq. (46)	$f_{numerical}(\eta)$	relative error $\varepsilon = \lvert f_{numerical}(\eta) - \bar{f}_{OHAM}(\eta) \rvert$
0	0.25	0.25	0.
4/5	0.6867557267	0.6867557534	$2.67 \cdot 10^{-8}$
8/5	0.7952832317	0.7952831549	$7.68 \cdot 10^{-8}$
12/5	0.8219262018	0.82192627455	$7.26 \cdot 10^{-8}$
16/5	0.8284614876	0.8284614200	$6.76 \cdot 10^{-8}$
4	0.8300643027	0.8300643067	$4.06 \cdot 10^{-9}$
24/5	0.8304574038	0.8304574546	$5.07 \cdot 10^{-8}$
28/5	0.8305539465	0.8305538923	$5.42 \cdot 10^{-8}$
32/5	0.8305776905	0.8305775608	$1.29 \cdot 10^{-7}$
36/5	0.8305834622	0.8305833895	$7.26 \cdot 10^{-8}$
8	0.8305847981	0.8305848529	$5.47 \cdot 10^{-8}$

Table 5
COMPARISON BETWEEN THE FIRST-ORDER APPROXIMATE SOLUTION $\bar{f}$ GIVEN BY EQ. (46) WITH NUMERICAL SOLUTION

η	$\bar{f}'_{OHAM}(\eta)$ from Eq. (46)	$f'_{numerical}(\eta)$	relative error $\varepsilon = \lvert f'_{numerical}(\eta) - \bar{f}'_{OHAM}(\eta) \rvert$
0	1.	0.9999999999	$1.11 \cdot 10^{-16}$
4/5	0.2523003767	0.2523010529	$6.76 \cdot 10^{-7}$
8/5	0.0620107099	0.0620107896	$7.97 \cdot 10^{-8}$
12/5	0.0152115047	0.0152114260	$7.87 \cdot 10^{-8}$
16/5	0.0037310135	0.0037309432	$7.03 \cdot 10^{-8}$
4	0.0009149428	0.0009150971	$1.54 \cdot 10^{-7}$
24/5	0.0002245184	0.0002244562	$6.21 \cdot 10^{-8}$
28/5	0.0000552208	0.0000550683	$1.52 \cdot 10^{-7}$
32/5	0.0000135435	0.0000135305	$1.29 \cdot 10^{-8}$
36/5	$3.21 \cdot 10^{-6}$	$3.35 \cdot 10^{-6}$	$1.41 \cdot 10^{-7}$
8	$7.27 \cdot 10^{-7}$	$8.74 \cdot 10^{-7}$	$1.46 \cdot 10^{-7}$

Table 6
COMPARISON BETWEEN THE DERIVATIVE $\overline{f'}$ OBTAINED FROM EQ. (46) WITH NUMERICAL SOLUTION

From the above tables, it can be seen that the approximate solutions obtained by the proposed procedure are nearly identical with numerical solutions.

Also, we note that the accuracy of the obtained results is growing along with increasing the number of the parameters C_i in the auxiliary functions H_i.

Conclusions

In the present paper, we found analytic approximate solutions for the upper-convected Maxwell fluid over a porous stretching plate applying the Optimal Homotopy Asymptotic Method. In the construction of the OHAM appear some distinctive concepts as: the auxiliary functions H_1, H_2, ..., the linear operator *L* and several optimal convergence-control parameters C_1,C_2, ... which ensure a fast convergence of the all solutions. The obtained results by OHAM are of the exceptional accuracy using only one iteration. Our procedure provides us with a simple and rigorous way to control and adjust the convergence of the solutions by means the auxiliary functions. The capital strength of the OHAM is its fast convergence, which proves that this method is very efficient in practice.

OHAM is an adequate approach for the practical interests like as the flow of the polymeric materials in the injection procedure.

In the boundary conditions imposed in our problem, OHAM can be usefully alternative in the searching and modeling of the polymeric viscous flow fluids.

References

1. B. C. SAKIADIS, Boundary layer behaviour an continuous solid surface I: Boundary layer equations for two dimensional and axisymmetric flow, AIChEJ, 1961, 26–28.
2. N. PHAN-THIEU, Plane and axisymmetric stagnation flow of a Maxwellian fluid, Rheol. Acta, 22, 1983, 127–130.
3. R. ZHENG, N. PHAN-THIEU, R. I. TANNER On the flow past a sphere in a cylindrical tube: Limiting Weissenberg number, J. Non-Newtonian Fluid Mech., 36, 1990, 27–49.
4. K. SADEGHY, A. H. NAJAFI, M. SAFFARIPOUR, Sakiadis flow of an upper-convected Maxwell fluid, Int. J. Non-Linear Mech., 40, 2005, 1220–1228.

5. T. HAYAT, M. SAJID, Homotopy analysis method of MHD boundary layer flow of an upper-convected Maxwell fluid, Int. J. of Eng. Science, 45, 2007, 393–401.
6. B. SAHOO, Effect of slip viscous dissipation and Joule heating on the MHD flow and heat transfer of a second grade fluid past a radially stretching sheet, Applied Mathematics and Mechanics, Engl. Ed. 31(2), 2010, 159–173.
7. V. MARINCA, N. HERISANU, Nonlinear Dynamical Systems in Engineering - Some Approximate Approaches, Springer Verlag, Heidelberg, 2011.
8. V. MARINCA, N. HERISANU, I. NEMES, Optimal homotopy asymptotic method with application to thin film flow, Cent. Eur. J. Phys., 6, 2008, 648–653.
9. V. MARINCA, N. HERISANU, C. BOTA, B. MARINCA, An optimal homotopy asymptotic method applied to the steady flow of a fourth grade fluid past a porous plate, Applied Mathematics Letters, 22, 2009, 245–251.
10. R.-D. ENE, V. MARINCA, R. NEGREA, B. C RUNTU, Optimal homotopy asymptotic method for solving a nonlinear problem in elasticity, 14th SYNASC (2012), DOI: 10.1109.SYNASC.2012.12.
11. R.-D. ENE, V. MARINCA, B. MARINCA, Approximate analytical solutions to nonlinear vibrations of a thin elastic plate, Acoustic & Vibration of Mechanical Structures, 430, 2013, 40–44.
12. L. ELSGOLTS, Differential Equations and the Calculus of Variations, Moscow, Mir Publishers, 1980

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